PLANE AND AXIALLY SYMMETRICAL AUTOMODEL (SIMILARITY) PROBLEMS OF PENETRATION AND OF STREAM IMPACT

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This article describes an analysis of the general problem of the penetration of a solid cone into a coaxial conical region which encloses ideal incompressible weightless fluid. The cone velocity is a power function of time. Both plane and axially symmetrical cases are reviewed.

Cone penetration at constant velocity is a special case of this problem [1-5]. This problem also embraces that of a transient model of the cumulative explosion of a missile in a conical envelope.

An approximate method of calculating the resistance and the velocity profile along the free surface is proposed (in particular, calculation of the velocity of the spray or the height of the accumulated wave). The method is applicable to any geometry.

1. Basis of Problem. 1. We begin by discussing the penetration of ideal weightless fluid by a wedge. At the start ($t = 0$), the fluid is at rest and occupies a solid angle of coaxial with the wedge (Fig.1). It is assumed that the wedge and the region r occupied by the fluid share a vertical axis of symmetry. The penetration velocity of the wedge is vertically downwards and is considered to be known. It is a power function of time, represented by

$$\mathbf{V} = -ct^{\gamma}\mathbf{y}^{\circ}$$

The fluid begins to move as a result of the pressure of the wedge. Because there are no mass forces, this motion will have a potential $\phi(x, y, t)$ which should satisfy the following boundary value problem (see Fig. 2 for notation);

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \qquad \text{in region } r$$

$$\frac{\partial\varphi}{\partial n} = V\cos\alpha \qquad \text{on } B'AB \tag{1.1}$$
$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] = 0 \qquad \text{on } S$$

The normal here is assumed to be the inner one with respect to the fluid, S is the free surface $y = \zeta(x, t)$. This is unknown at the outset and is determined from the kinematic condition

$$\frac{d\zeta}{dt} = \left(\frac{\partial\varphi}{\partial y}\right)_{y=\zeta} \tag{1.2}$$

The following condition of regularity

$$\lim \nabla\varphi = 0 \qquad \text{for } x^2 + y^2 \to \infty \tag{1.3}$$

and initial conditions should also be satisfied

$$\varphi(0, x, y) = 0, \qquad \zeta(0, x) = -|x|\operatorname{ctg}\beta \tag{1.4}$$

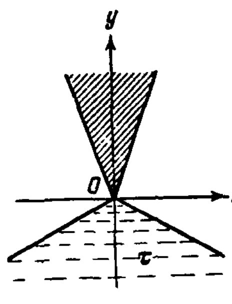

Fig. 1.

2. The wedge penetration problem will be a special case of this problem when the angle $\beta = \pi/2$. The Wagner-Sedov constant penetration velocity problem can be obtained from the present one by putting the index $\gamma = 0$.

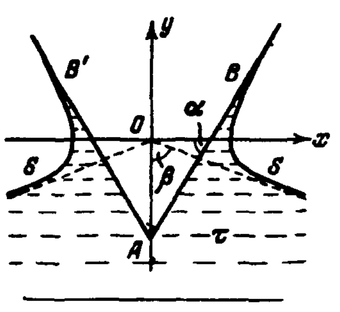

Fig. 2.

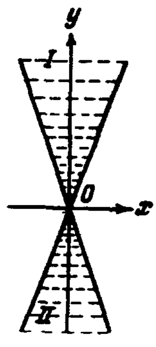

Fig. 3.

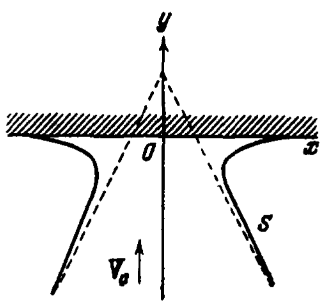

Fig. 4.

The problem of symmetrical stream impact is also a particular case if we assume that streams I and II (Fig. 3) are geometrically similar and are symmetrically located with respect to the x and y axes. At the instant of starting, too, they should share point 0, whilst at infinity the velocities of the fluid particles are equal in magnitude, opposite in sign, and vary according to a power law.

Because of symmetry the x axis will be a streamline. If we introduce a system of coordinates associated with a plane and put $\alpha = 0$, the above problem becomes that of lateral flow of fluid against a plane (Fig. 4).

Note. It is obvious that the problem of lateral flow of a fluid wedge under gravity, from an initial state of rest, can be reduced to the above problem. In that case we should put $\alpha = 0$ and $\gamma = 1$.

3. Problem (1.1) to (1.4) is a similarity or "automodel" one. Because of this, the solution will depend on two dimensionless groups of the variables x, y and t.

Owing to symmetry about the y axis it is sufficient to study the flow on the right hand side of the y axis.

Introduce the dimensionless variables ξ and η by means of the following transformations

$$\xi = \frac{x}{ct^{\gamma+1}} \sin\alpha - \frac{y}{ct^{\gamma+1}} \cos\alpha - \cos\alpha \qquad (1.5)$$

$$\eta = \frac{x}{ct^{\gamma+1}} \cos\alpha + \frac{y}{ct^{\gamma+1}} \sin\alpha + \sin\alpha$$

The velocity potential $\phi(x, y, t)$ and the equation of the free boundary $\xi(x, t)$ can be represented thus

$$\varphi = c^2 t^{2\gamma+1} \Phi(\xi, \eta), \qquad y = \zeta(x, t) = ct^{\gamma+1}\{f(\xi)\sin\alpha - \xi\cos\alpha - 1\} \qquad (1.6)$$

where Φ and f are the required non-dimensional functions.

Our chosen system of coordinates ξ, η is convenient because the expression for the free boundary $\eta = f(\xi)$ is a single-valued function of the one variable ξ.

To the region τ in variables ξ, η, whose shape varies in time, there corresponds a completely determinate invariable region $\bar{\tau}$ (Fig. 5).

The function $\Phi(\xi, \eta)$ in the $\bar{\tau}$ region will be a harmonic in variables ξ and η.

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} = 0$$

The equation of the sides of the wedge in this coordinate system will be $\xi = 0$. Let us find the boundary conditions for function Φ.

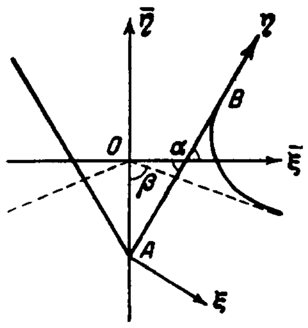

Fig. 5.

To do this, we first of all work out the differentials of the function $\phi(x, y, t)$;

$$\frac{\partial\varphi}{\partial t} = c^2 t^{2\gamma}\left\{(2\gamma+1)\Phi - (\gamma+1)(\xi+\cos\alpha)\frac{\partial\Phi}{\partial\xi} - (\gamma+1)(\eta-\sin\alpha)\frac{\partial\Phi}{\partial\eta}\right\}$$

$$\frac{\partial\varphi}{\partial x} = ct^{\gamma}\left(\frac{\partial\Phi}{\partial\xi}\sin\alpha + \frac{\partial\Phi}{\partial\eta}\cos\alpha\right)$$

$$\frac{\partial\varphi}{\partial y} = ct^{\gamma}\left(-\frac{\partial\Phi}{\partial\xi}\cos\alpha + \frac{\partial\Phi}{\partial\eta}\sin\alpha\right)$$

Let $\mathbf{n}^0$ denote the vector of the normal external to the wedge surface. We then have

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x}\cos(n, x) + \frac{\partial\varphi}{\partial y}\cos(n, y) = ct^{\gamma}\left\{\left(\frac{\partial\Phi}{\partial\xi}\sin\alpha + \frac{\partial\Phi}{\partial\eta}\cos\alpha\right)\sin\alpha + \right.$$

$$\left. + \left(-\frac{\partial\varphi}{\partial\xi}\cos\alpha + \frac{\partial\Phi}{\partial\eta}\sin\alpha\right)(-\cos\alpha)\right\} = ct^{\gamma}\frac{\partial\Phi}{\partial\xi}$$

Now put this expression into the first of the conditions (1.1) and we find that along $B'AB$ the function Φ satisfies the condition

$$\frac{\partial\Phi}{\partial\xi} = \cos\alpha \tag{1.7}$$

The assymptotic relation (1.3) will apparently remain valid, i.e.

$$\lim \nabla\Phi = 0 \qquad \text{for } \xi^2+\eta^2\to\infty \tag{1.8}$$

Moreover the constant pressure condition (the second in (1.1)) takes the following form;

$$(2\gamma+1)\Phi - (\gamma+1)(\xi+\cos\alpha)\frac{\partial\Phi}{\partial\xi} - (\gamma+1)(\eta-\sin\alpha)\frac{\partial\Phi}{\partial\eta} +$$

$$+ \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] = 0 \tag{1.9}$$

We now transform the kinematic condition

$$\frac{d\zeta}{dt} = \left(\frac{\partial\varphi}{\partial y}\right)_{y=\zeta}$$

In dimensionless variables it will be:

$$\left(\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial\xi}\frac{d\xi}{dt}\right)_{\eta=f(\xi)} = \left\{ct^{\gamma}\left(-\frac{\partial\Phi}{\partial\xi}\cos\alpha + \frac{\partial\Phi}{\partial\eta}\sin\alpha\right)\right\}_{\eta=f(\xi)} \qquad (1.10)$$

Now we do some auxiliary calculation

$$\frac{\partial\zeta}{\partial t} = (\gamma+1)\,ct^{\gamma}\,\{f(\xi)\sin\alpha - \xi\cos\alpha - 1\}$$

$$\frac{\partial\zeta}{\partial\xi} = ct^{\gamma+1}\,\{f'(\xi)\sin\alpha - \cos\alpha\}$$

$$\frac{d\xi}{dt} = \frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\frac{dx}{dt} + \frac{\partial\xi}{\partial y}\frac{dy}{dt} = \frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\xi}{\partial y}\frac{\partial\varphi}{\partial y}$$

$$\frac{\partial\xi}{\partial t} = -(\gamma+1)\frac{x}{ct^{\gamma+2}}\sin\alpha + (\gamma+1)\frac{y}{ct^{\gamma+2}}\cos\alpha = -(\gamma+1)\frac{1}{t}(\xi + \cos\alpha)$$

$$\frac{\partial\xi}{\partial x} = \frac{\sin\alpha}{ct^{\gamma+1}} \qquad \frac{\partial\xi}{\partial y} = -\frac{\cos\alpha}{ct^{\gamma+1}}$$

$$\frac{d\xi}{dt} = -(\gamma+1)\frac{1}{t}(\xi+\cos\alpha) + \frac{\sin\alpha}{ct^{\gamma+1}}\,ct^{\gamma}\left(\frac{\partial\Phi}{\partial\xi}\sin\alpha + \frac{\partial\Phi}{\partial\eta}\cos\alpha\right) -$$

$$-\frac{\cos\alpha}{ct^{\gamma+1}}\,ct^{\gamma}\left(-\frac{\partial\Phi}{\partial\xi}\cos\alpha + \frac{\partial\Phi}{\partial\eta}\sin\alpha\right) = -(\gamma+1)\frac{1}{t}(\xi+\cos\alpha) + \frac{1}{t}\frac{\partial\Phi}{\partial\xi}$$

Both here, and in the expressions to follow, a dash denotes a derivative with respect to ξ. Insert these expressions into condition (1.10) and, after some rearrangement, we bring it to the following form

$$(\gamma+1)\{f - \sin\alpha - f'\xi - f'\cos\alpha\} + \frac{\partial\Phi}{\partial\xi}f' = \frac{\partial\Phi}{\partial\eta}$$

Now introduce the expressions for velocity

$$\frac{\partial\Phi}{\partial\xi} = u, \qquad \frac{\partial\Phi}{\partial\eta} = v$$

and, finally, the kinematic condition along the free boundary can be written thus

$$v = (f - \sin\alpha)(\gamma+1) - f'\{(\gamma+1)(\xi+\cos\alpha) - u\} \qquad (1.11)$$

4. One important thing should be mentioned here for future reference, namely, if the equation of the free surface is known, i.e. if function $f(\xi)$ is known (for instance from experiment), the velocity distribution u, v along the free surface reduces to solving an ordinary nonlinear differential equation. To obtain this equation we must eliminate v from equations (1.9) and (1.11).

In equation (1.11) all the functions depend on ξ. On differentiating it we obtain

$$v' = f'u' - f''\{(\gamma+1)(\xi+\cos\alpha) - u\} \tag{1.12}$$

Moreover, bearing in mind that along the free boundary

$$\frac{d\Phi}{d\xi} = u + f'v$$

we take the total derivative of the dynamic relation (1.9);

$$\gamma u + \gamma v f' + (\gamma+1)\left\{u'\left(\xi+\cos\alpha - \frac{u}{\gamma+1}\right) + v'\left(f - \sin\alpha - \frac{v}{\gamma+1}\right)\right\} = 0$$

Replacing v' by its expression (1.12), we obtain

$$\gamma u + \gamma v f' - (\gamma+1)\left\{u'\left(\xi+\cos\alpha - \frac{u}{\gamma+1}\right) + [f'u' - \right.$$

$$\left. - f''\langle(\gamma+1)(\xi+\cos\alpha - u)\rangle]\left(f - \sin\alpha - \frac{v}{\gamma+1}\right)\right\} = 0$$

v can be eliminated from this equation by means of (1.11). After some obvious rearrangement we arrive at

$$[(\gamma+1)(\xi+\cos\alpha) - u]\,[u'(1+f'^2) - f'f''\{(\gamma+1)(\xi+\cos\alpha) - u\} +$$

$$+ \gamma f'^2] = \gamma u + \gamma(\gamma+1) f'(f - \sin\alpha) \tag{1.13}$$

Equation (1.13) is an ordinary first-order nonlinear differential equation. We must solve this, putting $\xi = \infty$. The solution should also satisfy the condition (1.7), i.e. $u(0) = \cos\alpha$. We thus arrive at the boundary problem for a first-order equation. In general the problem cannot be solved. We will see below that a further condition can be used to get a reasonable shape for the free surface.

Equation (1.13) can be simplified if $\gamma = 0$ (velocity of wedge or of the stream constant)

$$(\xi+\cos\alpha - u)\,[u'(1+f'^2) - f''f'(\xi+\cos\alpha - u)] = 0 \tag{1.14}$$

At the surface of the wedge ($\xi = 0$) the normal wedge velocity component is $u = \cos\alpha$. From physical considerations, it is evident, too, that $u < \cos\alpha$ for any value of $\xi > 0$. Therefore the multiplier for no values $\xi \neq 0$. Furthermore, from equation (1.14) we have

$$\xi + \cos\alpha - u \neq 0$$

$$u' = \frac{f'f''(\xi+\cos\alpha - u)}{1+f'^2} \tag{1.15}$$

If $f(\xi)$ is already given this becomes an ordinary linear differential equation in u. Thus if $\gamma = 0$ it can be solved by quadrature. Moreover, if we insert the solution so found into kinematic condition, (1.11) we determine the velocity v without further quadratures. The case $\gamma = 0$, it should be mentioned, does not exhaust all the values of parameter γ for which equation (1.13) can be solved in quadratures.

To explain this circumstance more fully, let us put equation (1.13) into a slightly altered form. Solving for u', we get (1.16)

$$u' = \frac{f'f''[(\gamma+1)(\xi+\cos\alpha)-u]}{1+f'^2} - \frac{\gamma f'^2}{1+f'^2} + \frac{\gamma u + (\gamma+1)\gamma(f-\sin\alpha)f'}{(1+f'^2)[(\gamma+1)(\xi+\cos\alpha)-u]}$$

Now introduce the new variable $y = (\gamma + 1)(\xi + \cos\alpha) - u$. Then equation (1.16) becomes:

$$y' = -\frac{f'f''}{1+f'^2}y + (2\gamma+1) - \frac{B}{y} \tag{1.17}$$

where

$$B = \frac{\gamma(\gamma+1)[f'(f-\sin\alpha)+\xi+\cos\alpha]}{1+f'^2}$$

If $\gamma = 0$, then $B = 0$, and we then have

$$y' = -\frac{f'f''}{1+f'^2}y + 1$$

This equation corresponds to (1.15). Equation (1.17) reduces to a linear one in the further two cases:

1. When $\gamma = -1$. We have $B = 0$ and therefore

$$y' = -\frac{f'f''}{1+f'^2}y - 1$$

2. When $\gamma = -1/2$. By making the change of variable $y^2 = z$ we arrive at a first-order linear equation

$$z' = -\frac{f'f''}{2(1+f'^2)}z - \frac{B}{2}$$

2. Determination of Resultant Fluid Pressure. We will work out the drag of the wedge. Denote the momentum within volume τ by $\mathbf{K}$, then:

$$\frac{d\mathbf{K}}{dt} + \mathbf{F} = 0 \tag{2.1}$$

where $\mathbf{F}$ is the vector resultant force of the fluid on the wedge. Because of symmetry its projection on the x axis will be zero. For the other projection we have

$$F_y = -\frac{d\mathbf{K}_y}{dt} \tag{2.2}$$

Work out K_y

$$K_y = \rho\int\limits_\tau v_y\,d\tau = \rho\int\limits_\tau \frac{\partial\varphi}{\partial y}\,d\tau \tag{2.3}$$

Introducing nondimensional variables

$$\bar\xi = \frac{x}{ct^{\gamma+1}}, \qquad \bar\eta = \frac{y}{ct^{\gamma+1}}$$

we get

$$K_y = \rho c^3 t^{3\gamma+2} K^* \qquad \left(K^* = \int\limits_{\bar\tau} \frac{\partial\Phi}{\partial\bar\eta}\,d\bar\xi\,d\bar\eta\right) \tag{2.4}$$

Region $\bar\tau$ in the variables $\bar\xi$ and $\bar\eta$ is fixed, and therefore the drag force will be

$$F_y = -\frac{dK_y}{dt} = -\rho\,(3\gamma+2)\,c^3 t^{3\gamma+1} K^* \tag{2.5}$$

Now transform the integral within expression (2.5) by Green's Theorem

$$K^* = \int\limits_{\bar\tau} \nabla\Phi\cdot\nabla\eta\,d\bar\xi\,d\bar\eta = \int\limits_{\bar S} \bar\eta\,\frac{\partial\Phi}{\partial n}\,dS$$

Here $\bar S$ is the surface enclosing the volume $\bar\tau$. First of all let us assume that it consists of free surfaces S_1 and S_2, the sides of the wedge σ_1 and σ_2 and an arc of sufficiently large radius R (Fig. 6). The normal here is taken external with respect to the fluid.

$$K^* = \int\limits_{S_1+S_2} \bar\eta\,\frac{\partial\Phi}{\partial n}\,dS + \int\limits_{\sigma_1+\sigma_2} \bar\eta\,\frac{\partial\Phi}{\partial n}\,dS + \int\limits_{S_R} \bar\eta\,\frac{\partial\Phi}{\partial n}\,dS$$

From symmetry, this expression can be put;

$$K^* = 2J_1 + 2J_2 + J_3$$

where

$$J_1 = \int\limits_{S_1} \bar\eta\,\frac{\partial\Phi}{\partial n}\,dS, \qquad J_2 = \int\limits_{\sigma_1} \eta\,\frac{\partial\Phi}{\partial n}\,dS, \qquad J_3 = \int\limits_{S_R} \bar\eta\,\frac{\partial\Phi}{\partial n}\,dS \tag{2.6}$$

Now let us deal with the integral

$$J_3 = \int\limits_{S_R} \Phi\eta^\circ\,dS = -\int\limits_{S_R} \Phi\cos\theta\,dS$$

Here $\mathbf{n}$ is the normal external to the fluid, S is external to the wedge, η° is unit vector along axis $\bar\eta$. Evidently $\nabla\bar\eta = \eta^\circ$.

Now let us examine the Laurent expansion of the complex potential $W = \Phi + i\Psi$ in the neighbourhood of point $\bar\zeta = \bar\xi + i\bar\eta = \infty$. As the fluid at infinity is at rest, $(dW/dz)_\infty = 0$, so that the expansion $W(z)$ will not contain positive powers of z. Thus

$$W = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

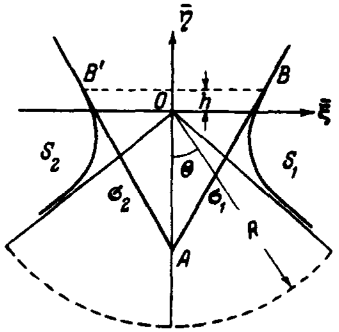

Fig. 6.

We can put $a_0 = 0$ without losing generality. (At the starting instant the fluid is at rest; $\phi \equiv 0$ for $t = 0$; as at infinity the fluid is at rest, for any value of t we have $(\phi)_\infty \equiv 0$, hence $(W)_\infty = 0$). Coefficient a_1 is zero likewise because owing to symmetry there is no flow through the $\bar\eta$ axis, and therefore the circulation is zero. Thus the potential $\Phi = \mathrm{Re}\, W$ can be represented by the following formula;

$$\Phi(r, \theta) = \frac{\chi(r, \theta)}{r^2}$$

where function $\chi(r, \theta)$ is limited for any values of θ for $r \to \infty$. Therefore in equation (2.6)

$$\lim J_3 = 0 \qquad \text{for } r \to \infty$$

For the integral J_2 in (2.6) we have

$$2J_2 = -2\int_{\sigma_1} \bar\eta \cos\alpha \, dS = 2\int_{\sigma_1} \boldsymbol{\eta}\cdot \mathbf{dS}$$

Denote the volume of wedge immersed in the fluid by τ_1:

$$\tau_1 = \int_{\tau_1} d\tau = \int_{\tau_1} \mathrm{div}\, \bar\eta \, d\tau = 2\int_{\sigma_1} \boldsymbol{\eta}\cdot \mathbf{dS} + \int_{BB'} \boldsymbol{\eta}\cdot \mathbf{dS}$$

Along BB' $\bar\eta = h =$ const. Thus, if L is taken as the length of BB', then

$$2\int_{\sigma_1} \boldsymbol{\eta}\cdot \mathbf{dS} = \tau_1 - hL$$

However, as $OA = 1$, from Fig. 6, we find

$$L = 2(h+1)\operatorname{ctg}\alpha$$

And thus

$$J_2 = \tau_1 - 2h(h+1)\operatorname{ctg}\alpha = (1-h^2)\operatorname{ctg}\alpha$$

because

$$\tau_1 = (h+1)^2 \operatorname{ctg} \alpha$$

We will write the integral in the coordinate system ξ, η (Fig. 5). Making use of the fact that

$$\bar{\eta} = f(\xi) \sin \alpha - \xi \cos \alpha - 1, \qquad \frac{\partial \Phi}{\partial n} = \frac{1}{\sqrt{1+f'^2}} (v - f'u)$$

$$dS = \sqrt{1+f'^2}\, d\xi$$

where $\eta = f(\xi)$ is the shape of the free fluid surface in this system of coordinates, u and v, normal and tangential velocities respectively of the fluid with respect to the wedge, it is possible to express it in this form

$$\int_{S_1} \bar{\eta} \frac{\partial \Phi}{\partial n} dS = \int_0^\infty [f(\xi) \sin \alpha - \xi \cos \alpha - 1][v - f'(\xi) u]\, d\xi$$

And finally we obtain the following formula for K^*;

$$K^* = (1-h^2) \operatorname{ctg} \alpha + 2 \int_0^\infty [f(\xi) \sin \alpha - \xi \cos \alpha - 1][v - f'(\xi) u]\, d\xi \quad (2.7)$$

where

$$h = f(0) \sin \alpha - 1$$

Therefore if the free surface is known, the problem of finding the drag force reduces to the solution of an ordinary differential equation and to quadratures.

3. Three-Dimensional Problems. 1. We will now deal with two three-dimensional problems; namely, that of axially symmetric penetration of an infinite cone into an ideal incompressible fluid, and the problem of the axially symmetric cumulative jet. To find the velocity potential $\phi(x,y,t)$ we will solve the following boundary problem (notation the same as in Fig. 2. except that x is now a radius, the distance from the axis of symmetry y);

$$\begin{aligned} &\frac{\partial^2 \varphi}{\partial y^2} + \frac{1}{x} \frac{\partial}{\partial x}\left(x \frac{\partial \varphi}{\partial x}\right) = 0 && \text{in the region of } \tau \\ &\frac{\partial \varphi}{\partial n} = V \cos \alpha && \text{for } B'AB \\ &\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] = 0 && \text{for } S \end{aligned} \qquad (3.1)$$

As before, we here take the normal positive inward with respect to the fluid, S is the free surface $y = \zeta(x, t)$ which is determined from the kinematic condition

$$\frac{d\zeta}{dt} = \left(\frac{\partial\varphi}{\partial y}\right)_{y=\zeta} \qquad (3.2)$$

Additionally, the condition of regularity must be fulfilled

$$\lim \nabla\varphi = 0 \qquad \text{for } y^2 + x^2 \to \infty \qquad (3.3)$$

and the initial condition

$$\varphi(x, y, 0) = 0, \qquad \zeta(x, 0) = x \operatorname{ctg} \beta \qquad (3.4)$$

In view of symmetry about axis Oy it is sufficient to deal only with the flow in any semi-plane meridian section.

Introduce dimensionless coordinates ξ and η (Fig. 5), connected with x, y and t by relations (1.5) and the velocity potential $\phi(x, y, t)$, and the equation of the free surface $y = \zeta(x, t)$ will be put in form (1.6). The boundary conditions on the cone generator, at infinity and on the free surface of the fluid in ξ, η coordinates, do not change their form as compared with the plane case and can be described by formulas (1.7), (1.8), (1.9) and (1.11) respectively, but the equation for potential $\Phi(\xi, \eta)$ will be different. After some rearrangement it can be put into this form

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} + \frac{1}{\xi\sin\alpha + \eta\cos\alpha}\left(\frac{\partial\Phi}{\partial\xi}\sin\alpha + \frac{\partial\Phi}{\partial\eta}\cos\alpha\right) = 0 \qquad (3.5)$$

The analysis carried out for the relation for the free surface (see 4, para. 1) is also valid for the case of axial symmetry.

2. We now work out the drag as the cone penetrates the fluid. From the theorem of momentum it follows that the force F_y, acting on the cone vertically, is;

$$F_y = dK_y / dt \qquad (3.6)$$

where K_y is the momentum component along the y axis of the matter enclosed in volume τ. Owing to symmetry, the other momentum components within the fluid are zero.

To work out K_y

$$K_y = \rho\int_\tau v_y d\tau = \rho\int_\tau \frac{\partial\varphi}{\partial y}\, d\tau \qquad (3.7)$$

we use nondimensional variables

$$\bar\xi = \frac{x}{ct^{\gamma+1}}, \quad \bar\eta = \frac{y}{ct^{\gamma+1}}, \quad \bar\zeta = \frac{z}{ct^{\gamma+1}} \qquad (3.8)$$

Then we have

$$\frac{\partial \varphi}{\partial y} = ct^{\gamma}\frac{\partial \Phi}{\partial \bar{\eta}}, \qquad d\tau = \frac{D(x, y, z)}{D(\bar{\xi}, \bar{\eta}, \bar{\zeta})}\, d\bar{\tau} = c^3 t^{3(\gamma+1)} d\bar{\tau} \tag{3.9}$$

$$K_y = \rho c^4 t^{4\gamma+3} K^*, \qquad K^* = \int_{\bar{\tau}} \frac{\partial \Phi}{\partial \bar{\eta}}\, d\bar{\tau} \tag{3.10}$$

Because the region $\bar{\tau}$ in the variables $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ is fixed, it follows that K^* is independent of time. Thus, for the drag force we have the expression

$$\dot{F}_y = -\rho(4\gamma + 3) c^4 t^{4\gamma+2} K^* \tag{3.11}$$

Rearrange the integral in expression (3.10);

$$K^* = \int_{\bar{\tau}} \nabla \Phi \cdot \nabla \bar{\eta}\, d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta} = \int_{\bar{S}} \bar{\eta}\, \frac{\partial \Phi}{\partial n}\, dS$$

where $\bar{S}$ is the surface enclosing the volume $\bar{\tau}$. Surface $\bar{S}$ can be considered as consisting of the wetted surface of the cone σ, free surface S and the surface of a sphere of sufficiently large radius S_R. Then for K^* we have the expression

$$K^* = \int_{s} \bar{\eta} \frac{\partial \Phi}{\partial n}\, dS + \int_{\sigma} \bar{\eta}\, \frac{\partial \Phi}{\partial n}\, dS + \int_{S_R} \bar{\eta}\, \frac{\partial \Phi}{\partial n}\, dS \tag{3.12}$$

After calculation similar to that carried out for the plane problem, we finally arrive at the following formula for K^*

$$K^* = \frac{1}{3}\pi \operatorname{ctg}^2 \alpha (1 + h)^2 (1 - 2h) + \tag{3.13}$$

$$+ 2\pi \int_0^\infty [f(\xi)\sin\alpha - \xi\cos\alpha - 1]\, [f(\xi)\cos\alpha + \xi\sin\alpha]\, [v - f'(\xi)\, u]\, d\xi$$

where $\eta = f(\xi)$ is the shape of the free surface of the fluid in ξ, η coordinates; u, v are respectively velocities normal and tangential to the cone generator, $h = f(0)\sin\alpha - 1$. Now, using (3.11), we are able to determine the force acting on the penetrating cone if the form of the free surface of the fluid $\eta = f(\xi)$ is known.

4. Method of Numerical Calculation. To work out the approximate value of the force acting on the body, and the velocity distribution along the free boundary, it is sufficient to find the shape of the latter approximately. This can be done by the laws of conservation. In particular, if we use only the law of conservation of mass, the free surface can be approximated by the following expression

$$\eta = f(\xi) = a\xi + b + de^{-c\xi} \quad (c > 0) \tag{4.1}$$

As the fluid is at rest at infinity, the free boundary in (ξ, η) coordinates has the asymptote

$$\eta = \operatorname{tg}(\beta - \alpha)\xi + \frac{\sin\beta}{\cos(\beta - \alpha)}$$

where α is the angle between the side of the wedge (cone generator) and the Ox axis, and β is the half angle of the liquid wedge (cone) (Fig. 5).

Therefore the coefficients a and b will be

$$a = \operatorname{tg}(\beta - \alpha), \quad b = \frac{\sin\beta}{\cos(\beta - \alpha)} \tag{4.2}$$

It follows from the condition of incompressibility of the fluid, moreover, that the immersed volume of the cone equals that of the fluid displaced. This condition gives one of the equations for finding coefficients c and d;

in the plane case

$$\frac{d}{c} = \frac{1}{2}\,\frac{\sin\beta\cos\alpha}{\cos(\beta - \alpha)} \tag{4.3}$$

and for three dimensions

$$d^2 + 4\left(\frac{1}{c} + \cos\alpha\right)\frac{\sin\beta}{\cos\alpha\cos(\beta - \alpha)}\,d - \frac{2}{3}\,c\,\frac{\sin^2\beta\cos\alpha}{\cos^2(\beta - \alpha)} = 0 \tag{4.4}$$

The second equation, which connects these coefficients, can be obtained thus. To express the velocity u on the free fluid surface we have the ordinary first-order differential equation (1.15);

$$u' = \frac{f'f''(\xi + \cos\alpha - u)}{1 + f'^2} \tag{4.5}$$

The function u must here satisfy two conditions; that at infinity $u(\infty) = 0$ and that on the side of the wedge (cone generator) $u(0) = \cos\alpha$. We will use the first condition to find the arbitrary constant in solving equation (4.5). The condition on the side of the wedge (cone generator) will give the second tie-up between c and d, which, both in the plane and in the three-dimensional cases, reduces to one and the same transcendental equation in the following form

$$\left(\cos\alpha + \frac{1}{c}\right)\left(\sqrt{1 + y_0^2} - \sqrt{1 + y_\infty^2}\right) - \frac{1}{c}\left[y_\infty \ln\frac{\sqrt{1 + y_0^2} - y_0}{\sqrt{1 + y_\infty^2} - y_\infty} + \right.$$

$$\left. + \sqrt{1 + y_\infty^2}\ln\frac{1 + y_\infty y_0 + \sqrt{1 + y_\infty^2}\sqrt{1 + y_0^2}}{2(1 + y_\infty^2)}\right] - \sqrt{1 + y_0^2}\cos\alpha = 0 \tag{4.6}$$

where

$$y_0 = f'(0) = \operatorname{tg}(\beta - \alpha) - cd, \qquad y_\infty = f'(\infty) = \operatorname{tg}(\beta - \alpha)$$

In this manner the free boundary of the fluid is determined, and both kinematic and dynamic conditions are accurately fulfilled on it.

Now we can make use of the formulas deduced in the foregoing paragraphs. It is now possible to find approximately the value of the normal velocity at the free surface and also the total force acting on the immersed wedge (cone).

5. Some Results of Calculations. The high speed computer "Strela" was used for working out calculations in accordance with the above method.

These results are compared with known experimental results and some calculated results in the graphs shown below.

For wedges of angle α varying from 10° to 80° in steps of 10° the shape of the free surface (splash stream) was calculated (see Fig. 7). A general similarity can be seen in the relations between the distances from the cone vertex to the apices of the splash streams; with $\alpha \to \pi/2$ this distance tends to zero, and, conversely, with $\alpha \to 0$ it tends to infinity. Velocity profiles have been drawn along the free surface. It can be seen that the velocity at the tip of the splash stream increases rapidly with dicreasing wedge angle and tends to infinity when the wedge turns into a plate (Fig. 8).

Figure 9 gives calculated values of resultant thrust of the fluid on the wedge as a function of wedge angle α. Our results agree with experiment and with those worked out by other authors over a wide range of wedge

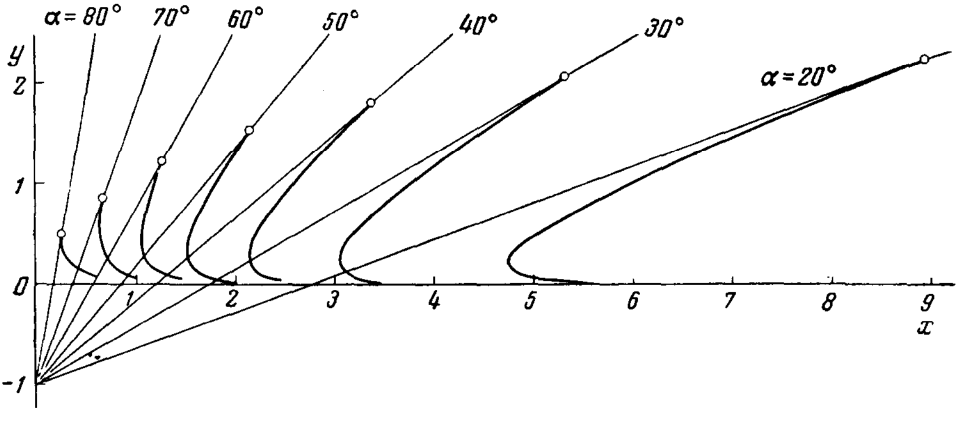

Fig. 7.

angles. They can be recommended for practical application for angles α between 20° and 90°. Curve 1 gives the reaction of the fluid on the wedge, curve 3 the resultant thrust on a cone, and curve 2 represents a development from the plate analogy. (1) is the load integral obtained in [4],

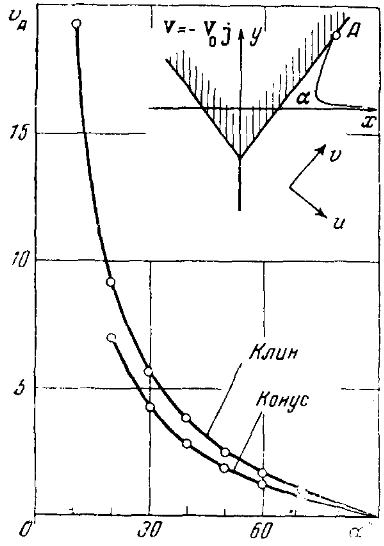

Fig. 8.

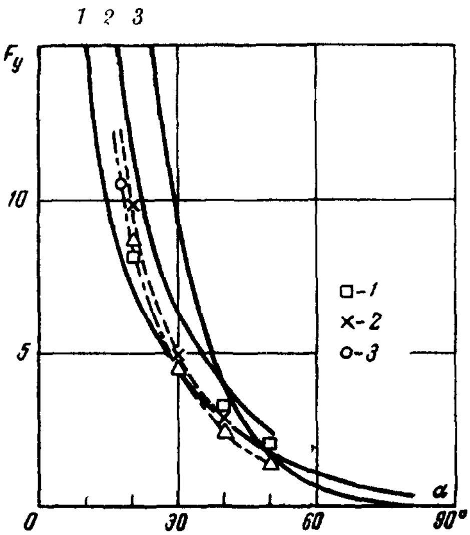

Fig. 9.

(2) is derived from a more accurate analysis [4], and (3) comes from Wagner's analysis for $\alpha = 18^{\circ}$.

The theory is open to improvement for small wedge angles. It may be assumed that the plate analogy might give a satisfactory result here.

The method of solving the penetration problem can be applied without any alteration to the problem of the lateral flow of a fluid wedge against a solid wall, with the condition that fluid particles at infinity go at constant velocity, and, at the initial instant, the fluid wedge touches the wall. Free surface shapes have been calculated, velocity profiles and the pressure of the wedge on the surface have been evaluated, all as a function of the half-angle of the wedge. (Fig. 10 shows the relation between the fluid thrust and the half-angle of a wedge).

Lavrentiev [6] proposed an explanation of the cumulative effect of an explosion with a conical envelope. According to Lavrentiev's model the movement of the envlope is similar to that of an impacting stream. Lavrentiev reviewed plane and axially symmetrical models, but the motion was considered to be stationary. Despite such an apparently rough method of approach, Lavrentiev's model has at least succeeded in giving some qualitative explanation of phenomena which seemed very complicated hitherto.

The approach to the solution of the stream impact problem in this work opens the way to a study of the more complicated mathematical model of the cumulative explosion, the model of transient accumulation of plane and conical charges.

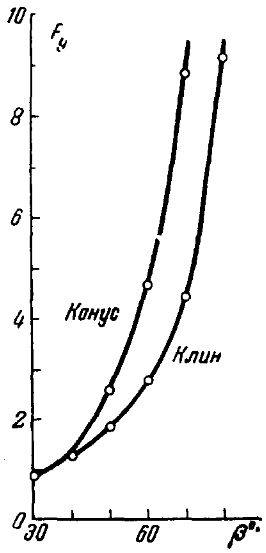

Fig. 10.

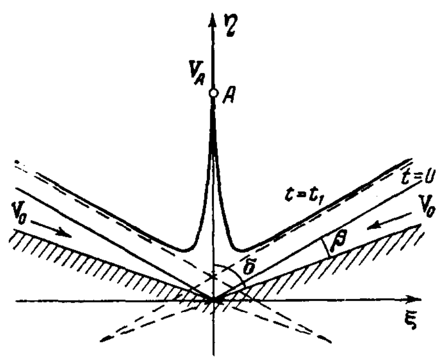

Fig. 11.

Suppose, at the initial instant, the fluid occupies a volume limited below by a solid conical surface with vertex angle of 2δ (Fig. 11), and a free boundary which also represents a cone, but with vertex angle $2(\delta - \beta)$. It is assumed that at the initial instant we have velocities V_a directed along the cone generator. It is easy to see that the problem put in this form reduces to the one discussed above. The forms of the free surfaces of cumulative streams have been worked out and the velocities along them have been calculated as functions of angles β and δ.

The table shows values of dimensionless velocities at the apices of cumulative streams for various stream "thicknesses" (angle β) and various envelope angles (δ). The table reveals that the apex velocity, which can

TABLE

δ	$\beta = 10°$	20°	30°	40°	50°	60°	70°	80°
90°	1.4627	1.9294	2.5032	3.2637	4.3550	6.1070	9.5046	19.4636
80°	1.7320	2.3161	3.0718	4,1487	5.8743	9.2197	19.0266	
70°	2.0197	2.7653	3.8032	5.4548	8.6496	18.0092		
60°	2.3337	3.3213	4.8577	7.8102	16.4418			
50°	2.6955	4.0932	6.7238	14.3708				
40°	3.1604	5.4158	11.8570					
30°	3.9006	8.9708						
20°	5.7776							

exceed V_0 by factors of ten, increases rapidly both with increased stream thickness and decreased envelope angle. The whole effect is a transient one, and therefore stationary models will not give such estimates.

It has been shown in 3 above that the solution of three-dimensional (axially symmetric) problems on cone penetration, on lateral flow of a fluid cone and impact of conical streams can be derived by direct analogy with plane problems. Free surface shapes and velocities along them have been calculated in terms of cone angles.

It is evident from this that the general pattern of velocity variation is similar to that of a wedge, but the changes take place over a region which lies closer to the surface of the body.

Figure 8 shows how the velocity at the nose of a splash stream varies. In the case of the cone the velocity increases less rapidly with reduced apex angle than in the case of the wedge. Figure 9 shows the relation between the reaction of the fluid on the cone for various cone angles α. On comparing these with the results for the wedge we see that for large values of α the curve for the cone lies below that of the wedge, but for values of α within the range 30° to 40° the cone displays a sudden increase in reaction, and the curve becomes much steeper than that of the wedge.

Figure 10 shows how the pressure of an outflowing fluid cone reacts on a conical surface for various angles of divergence of the latter, β. The pressure increases with increase in angle β more rapidly than in the case of plane flow.

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